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Almost Periodic Regularized Groups, Semigroups, and Cosine Functions*

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INTRODUCTION

Motivated by the abstract Cauchy problem and other interesting problems (e.g., differential operators [13, 16]), a generalization of C_0-semigroups, regularized semigroups, has received much attention since 1987. deLaubenfels has made a good summary on this theory in his recent monograph [10]. It is well known that paralleling the theory of semigroups of operators, one can develop the corresponding theory of cosine functions (see, e.g., [29]). Recently, the theory of regularized cosine functions has been developed by Kuo and Shaw [15], Lei and Zheng [17], Li and Shaw [19].

On the other hand, the almost periodicity of C_0-semigroups and C_0-groups have been studied systematically by Bart and Goldberg [3]. This paper also contained the results of weak and uniform almost periodicity. With their work, Baskakov [4], Cioranescu [7], Piskarev [25, 26], and others discussed the almost periodicity of strongly continuous cosine functions. Among others, Baskakov [4] and Piskarev [25] also discussed the almost periodicity of the corresponding sine functions. Before this, the periodic C_0-groups, which is a special case of the almost periodic C_0-group, had been discussed by Bart [2] and Da Prato [9] (see also [23, A-III-5]), where Da Prato considered the periodicity of regularized groups. A gap in [9] was pointed out and corrected by Bart [2] in the case of C_0-groups. The results of periodic strongly continuous cosine functions are due to Giusti [12],

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Lutz [21], Piskarev [26] *et al*. Furthermore, Da Prato [9] and Cioranescu [6] considered the periodicity of distribution groups.

In this paper, we discuss the almost periodicity of regularized groups, semigroups, cosine functions, and the corresponding sine functions. As a special case of the almost periodicity, the periodicity of these operator families is also considered. Our results generalize the corresponding results given by the above-mentioned papers. The paper is organized as follows.

In Section 1 we give some preliminaries, which contain the definitions and basic properties of regularized groups, semigroups, cosine functions, the corresponding sine functions, and almost periodic vector-valued functions. The definitions and some remarks of almost periodic and periodic strongly continuous operator families are also contained. Moreover, some notations used throughout this paper are stated at the beginning of this section.

In Section 2 our main purpose is to obtain a characterization of the generator of an almost periodic regularized group, which is a generalization of Theorem 2 in [3]. Among other results we give the conditions that the weak almost periodicity (resp. almost periodicity) of a regularized group implies its almost periodicity (resp. uniform almost periodicity) and give a generalization of Proposition 5 in [3].

In Section 3 we consider almost periodic regularized semigroups. First, we give the relationship between regularized semigroups and regularized groups. Next, we deduce from this that an (uniformly) almost periodic regularized semigroup can be extended to an (uniformly) almost periodic regularized group. Finally, combining this and the characterization of almost periodic regularized groups in Section 2, we characterize almost periodic regularized semigroups. The results in this section generalize the corresponding results in [24, Sect. 1.6], Theorem 8, 9 and Proposition 14 in [3].

In Section 4 we obtain the characterizations of almost periodic regularized cosine functions and the corresponding sine functions, the relationship between them, and the characterization of both of them being almost periodic. This section also contains the conditions under which their weak almost periodicity (resp. almost periodicity) implies their almost periodicity (resp. uniform almost periodicity). The results of this section generalize the corresponding results in [7] and [25], etc.

In Section 5 we characterize the periodicity of regularized groups, cosine functions, and the corresponding sine functions. In particular, the characterization of regularized cosine functions and the corresponding sine functions with the same period is shown. More information about periodic regularized groups and cosine functions are also obtained. These results generalize the corresponding results in [2, 21, 26].

In the last section we deal with a special case; i.e., the regular operator C (see Definition 1.1 and 1.3 below) is some power of the resolvent of the generator. In this case we generalize all spectral properties of the generator of almost periodic and periodic C_0-semigroups, strongly continuous cosine functions, and the corresponding sine functions (see[2, 3, 7, 6]). Finally, the results are applied to the distribution semigroup introduced by Lions [20], and a characterization of almost periodic tempered distribution semigroups is proven.

1. PRELIMINARIES

Throughout this paper, X is a complex Banach space. All operators are linear. $B(X)$ is the space of bounded operators on X. If A is an operator on X, then $D(A)$, $R(A)$, $\ker(A)$, $\rho(A)$, $\sigma(A)$, $P_\sigma(A)$, and $R(\lambda, A)$ ($\lambda \in \rho(A)$) denote, respectively, the domain, the range, the kernel, the resolvent set, the spectrum, the point spectrum, and the resolvent of A. Set X_a (resp. X_b) $= \overline{\operatorname{span}}\{x \in D(A);\ Ax = irx$ (resp. $= -r^2 x$) for some $r \in \mathbf{R}\}$. $C \in B(X)$ will be injective. Let $\rho_C(A) \equiv \{\lambda \in \mathbf{C};\ \lambda - A$ is injective and $R(C) \subset R(\lambda - A)\}$, $\sigma_C(A) \equiv \mathbf{C} \setminus \rho_C(A)$, and

$$R_C(\lambda, A) \equiv (\lambda - A)^{-1} C(\lambda \in \rho_C(A))$$

denote, respectively, the C-resolvent set, the C-spectrum, and the C-resolvent of A. Clearly, $P_\sigma(A) \subset \sigma_C(A)$. Moreover, set $\mathbf{J} = \mathbf{R}$ or $\mathbf{R}_+ (= [0, \infty))$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and $\mathbf{Z} = \mathbf{N}_0 \cup (-\mathbf{N})$.

We start with the definitions and properties of regularized semigroups and groups.

DEFINITION 1.1. If a strongly continuous operator family $T(t) \in B(X)$ ($t \in \mathbf{J}$) satisfies $T(0) = C$ and $T(t+s)C = T(t)T(s)$, $\forall t, s \in \mathbf{J}$, then $T(t)$ is called a C-regularized semigroup in the case $\mathbf{J} = \mathbf{R}_+$ and a C-regularized group in the case $\mathbf{J} = \mathbf{R}$. Their generator A is defined by

$$D(A) = \left\{ z \in X;\ \lim_{\mathbf{J} \ni t \to 0} \frac{1}{t}(T(t)x - x) \in R(C) \right\}$$

$$Ax = C^{-1} \lim_{\mathbf{J} \ni t \to 0} \frac{1}{t}(T(t)x - Cx) \qquad \text{for} \quad x \in D(A).$$

Clearly, when $C = I$, C-regularized semigroups (resp. C-regularized groups) coincide with C_0-semigroups (resp. C_0-groups).

The following are some properties of regularized semigroups and regularized groups in which the properties of regularized semigroups can be found in [10], while the properties of regularized groups can be obtained immediately by Theorem 3.1 in Section 3 and the corresponding properties of regularized semigroups.

LEMMA 1.2. *Let $T(t)$ $(t \in \mathbf{J})$ be a C-regularized semigroup or group with the generator A. Then*

(a) *A is closed and $R(C) \subset \overline{D(A)}$.*

(b) *$\forall f \in C^1(\mathbf{J}, X)$, $t \in \mathbf{J}$, $\int_0^t T(s)f(s)ds \in D(A)$ and*

$$A \int_0^t T(s)f(s)ds = T(t)f(t) - Cf(0) - \int_0^t T(s)f'(s)ds. \quad (1.1)$$

In particular, $\forall x \in X, t \in \mathbf{J}$, $\int_0^t T(s)xds \in D(A)$ and

$$A \int_0^t T(s)xds = T(t)x - Cx. \quad (1.2)$$

(c) *$\forall x \in D(A), t \in \mathbf{J}, T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ and*

$$\int_0^t T(s)Axds = T(t)x - Cx. \quad (1.3)$$

(d) *If $T(t)$ is uniformly bounded (i.e., $\sup_{t \in \mathbf{J}} \|T(t)\| < \infty$), then $\{\lambda \in \mathbf{C};$ Re $\lambda \in \mathbf{J} \setminus \{0\}\} \subset \rho_C(A)$ and*

$$R_C(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt, \qquad \forall x \in X, \qquad \text{Re}\ \lambda > 0. \quad (1.4)$$

The following definitions and properties of regularized cosine functions and the corresponding sine functions can be found in [15, 17].

DEFINITION 1.3. If a strongly continuous operator family $C(t) \in B(X)$ $(t \in \mathbf{R})$ satisfies $C(0) = C$ and $2C(t)C(s) = C(t+s)C + C(s-t)C$, $\forall t, s \in \mathbf{R}$, then $C(t)$ is called a C-regularized cosine function and $S(t)$ $(t \in \mathbf{R})$ the corresponding C-regularized sine function, where $S(t)x = \int_0^t C(s)xds$, $\forall x \in X, t \in \mathbf{R}$. The generator A of $C(t)$ is defined by

$$D(A) = \left\{ x \in X;\ \lim_{t \downarrow 0} \frac{1}{t^2}(C(t)x - Cx) \in R(C) \right\}$$

$$Ax = C^{-1} \lim_{t \downarrow 0} \frac{1}{t^2}(C(t)x - Cx) \qquad \text{for} \quad x \in D(A).$$

LEMMA 1.4. *Assume that A generates a C-regularized cosine function $C(t)$, and $S(t)$ is its corresponding C-regularized sine function. Then*

(a) *A is closed and $R(C) \subset \overline{D(A)}$.*

(b) *$C(t) = C(-t)$ and $S(-t) = -S(t)$, $\forall t \in \mathbf{R}$.*

(c) *$2C(t)S(s) = S(s+t)C + S(s-t)C$, $\forall t, s \in \mathbf{R}$.*

(d) *$\forall x \in X, t \in \mathbf{R}$, $S(\cdot)x \in X^1(\mathbf{R}, X)$, $\int_0^t S(s)xds \in D(A)$ and*

$$A \int_0^t S(s)xds = C(t)x - Cx. \tag{1.5}$$

(e) *$\forall x \in D(A)$, $t \in \mathbf{R}$, $C(t)x \in D(A)$, $AC(t)x = C(t)Ax$, and*

$$\int_0^t S(s)Axds = C(t)x - Cx. \tag{1.6}$$

(f) *If $C(t)$ is uniformly bounded* (*i.e.*, $\sup_{t \in \mathbf{R}} \|C(t)\| < \infty$), *then $\sigma_C(A) \subset (-\infty, 0]$ and*

$$\lambda R_C(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)xdt, \qquad \forall x \in X, \qquad \operatorname{Re} \lambda < 0. \tag{1.7}$$

DEFINITION 1.5. We say that $f \in C(\mathbf{J}, X)$ is almost periodic, written $f \in AP(\mathbf{J}, X)$, if $\forall \varepsilon > 0$, there exists $l > 0$ such that every subinternal of $\mathbf{J}$ of length l contains at least one τ satisfying $\|f(t+\tau) - f(t)\| \leq \varepsilon$, $\forall t \in \mathbf{J}$; $f \in C(\mathbf{J}, X)$ is weakly almost periodic, if $\langle x^*, f(\cdot)\rangle \in AP(\mathbf{J}, \mathbf{C})$, $\forall x^* \in X^*$; $\mathscr{F} \subset C(\mathbf{J}, X)$ is uniformly almost periodic, if $\forall \varepsilon > 0$ there exists $l > 0$ such that every subinternal of $\mathbf{J}$ of length l contains at least one τ satisfying $\|f(t+\tau) - f(t)\| \leq \varepsilon$, $\forall t \in \mathbf{J}, f \in \mathscr{F}$.

We collect some basic results of vector-valued almost periodic functions in the following lemma (see [18, 30]).

LEMMA 1.6. *Let $f \in AP(\mathbf{R}, X)$; then*

(a) *$f(t)$ is bounded, i.e.*, $\sup_{t \in \mathbf{R}} \|f(t)\| < \infty$;

(b) *if $g \in AP(\mathbf{R}, X)$, $h \in AP(\mathbf{R}, \mathbf{C})$, then $f + g, hf \in AP(\mathbf{R}, X)$*;

(c) *$\forall r, \alpha \in \mathbf{R}$, $a_r(f) \equiv \lim_{t \to \infty} 1/t \int_0^t e^{-irs} f(s)ds$ exists and $a_r(f) =$* $\lim_{t \to \infty} 1/t \int_\alpha^{t+\alpha} e^{-irs} f(s)ds$;

(d) *$\sigma(f) = \{r \in \mathbf{R}; a_r(f) \neq 0\}$ is at most countable*;

(e) *if $a_r(f) = 0$, $\forall r \in \mathbf{R}$, then $f(t) = 0$, $\forall t \in \mathbf{R}$*;

(f) *if $X \not\supset c_0$ and $g(t) \equiv \int_0^t f(s)ds (t \in \mathbf{R})$ is bounded, then $g \in$* $AP(\mathbf{R}, X)$;

(f) *if $f_n \in AP(\mathbf{R}, X)$ and f_n converge uniformly to f, then $f \in$* $AP(\mathbf{R}, X)$.

Reflexive and even weakly sequentially complete spaces do not contain c_0. Recall that a space does not contain c_0 if it does not contain subspaces isomorphic c_0.

Finally, we give the definitions of almost periodicity and periodicity of strongly continuous operator family.

DEFINITION 1.7. Let $F(t) \in B(X)$ $(t \in \mathbf{J})$ be a strongly continuous operator family. Then

(a) $F(t)$ is (weakly) almost periodic, if $F(\cdot)x$ is (weakly) almost periodic, $\forall x \in X$; $F(t)$ is uniformly almost periodic, if $\{F(\cdot)x;\ \|x\| \leq 1\}$ is uniformly almost periodic.

(b) $F(t)$ is periodic, if there exists $p > 0$ such that $F(t+p) = F(t)$, $\forall t \in \mathbf{J}$; $F(t)$ is strongly periodic, if for every $x \in X$, there exists $p > 0$ such that $F(t+p)x = F(t)x$, $\forall t \in \mathbf{J}$; $F(t)$ is weakly periodic, if for every $x \in X$ and $x^* \in X^*$, there exists $p > 0$ such that $\langle x^*, F(t+p)x \rangle = \langle x^*, F(t)x \rangle$, $\forall t \in \mathbf{J}$.

Obviously, the uniform almost periodicity (resp. almost periodicity) of $F(t)$ implies its almost periodicity (resp. weak almost periodicity). By the same method as in [2, proof of Theorem 2.1], one easily shows that the three notions of periodicity of $F(t)$ are equivalent. If $F(t)$ is a regularized semigroup, or group, or cosine function, or sine function, and if its generator is A with $\overline{D(A)} = X$, then the periodicity of $F(t)x$ $(\forall x \in D(A))$ implies the periodicity of $F(t)$. Moreover, if $F(t)$ $(t \geq 0)$ is a periodic regularized semigroup with period p, then it is clear that $T(t)$ may be extended to a periodic regularized group with period p by $T(-t) = T(np - t)$, $(n-1)p < t \leq np (n \in \mathbf{N})$.

2. ALMOST PERIODIC REGULARIZED GROUPS

In this section, $T(t)$ is a C-regularized group with the generator A except in Lemma 2.2. The following is the main result of this section.

THEOREM 2.1. *Let $\overline{R(C)} = X$. Then $T(t)$ is almost periodic if and only if $T(t)$ is uniformly bounded and $X_a = X$.*

Proof. Sufficiency. Since $X_a = X$, for any $x \in X, \varepsilon > 0$, there exist finitely many point, $r_k \in \mathbf{R}$ and $x_k \in \ker(ir_k - A)$ such that $\|x - \Sigma_k x_k\| \leq \varepsilon/M$, where $M = \sup_{t \in \mathbf{R}} \|T(t)\| < \infty$. But $T(\cdot)x_k = e^{ir_k \cdot} C x_k \in AP(\mathbf{R}, X)$. Hence $T(\cdot)x \in AP(\mathbf{R}, X)$ follows from $\|T(t)x - \Sigma_k T_k(t) x_k\| \leq \varepsilon$ and Lemma 1.6(b) and (g); i.e., $T(t)$ is almost periodic.

Necessity. From Lemma 1.6(a) and the uniform boundedness theorem it follows that $T(t)$ is uniformly bounded. Define

$$P_r x = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} T(s) x ds, \qquad \forall r \in \mathbf{R}, \qquad x \in X. \tag{2.1}$$

Then by Lemma 1.6(c), we know that $P_r x$ exists and satisfies

$$\begin{aligned}\frac{T(h) - C}{h} P_r x &= \frac{1}{h} \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} CT(h+s) x ds - \frac{1}{h} CP_r x \\ &= \frac{1}{h} e^{irh} CP_r x - \frac{1}{h} CP_r x \to irCP_r x (h \to 0),\end{aligned} \tag{2.2}$$

i.e., $P_r x \in D(A)$ and $AP_r x = irP_r x$. Hence $P_r x \in \ker(ir - A) \subset X_a$, $\forall r \in \mathbf{R}$, $x \in X$. If $x^* \in X^*$ such that $\langle x^*, x\rangle = 0$, $\forall x \in X_a$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} \langle x^*, T(s) x\rangle ds = \langle x^*, P_r x\rangle = 0, \qquad \forall r \in \mathbf{R}, \qquad x \in X.$$

But our assumption implies $\langle x^*, T(\cdot)x\rangle \in AP(\mathbf{R}, \mathbf{C})$. Thus by Lemma 1.6(e), $\langle x^*, T(t)x\rangle = 0$, $\forall t \in \mathbf{R}$, $x \in X$. In particular, $\langle x^*, Cx\rangle = 0, \forall x \in X$. Combining with $\overline{R(C)} = X$, it follows that $x^* = 0$, i.e., $x_a = X$. ∎

In order to characterize the generator of almost periodic regularized groups, we need the following result.

Lemma 2.2. *Let $\overline{R(C)} = X$, $M > 0$, and $\omega \in \mathbf{R}$. Then A generates a C-regularized semigroup $T(t)$ $(t \geq 0)$ with $\|T(t)\| \leq Me^{\omega t} (t \geq 0)$ if and only if $\overline{D(A)} = X$, $A = C^{-1}AC$, $(\omega, \infty) \subset \rho_C(A)$, $R(C) \subset R((\lambda - A)^n)$, and $\|(\lambda - \omega)^n (\lambda - A)^{-n} C\| \leq M, \forall \lambda > \omega, n \in N$.*

Proof. By Corollary 2 and its Remark in [22], we only need to show that the sufficiency condition of the theorem implies the closedness of A. In fact, if $D(A) \ni x_n \to x$ and $Ax_n \to y$, then $(\lambda_0 - A)^{-1} C(\lambda_0 - A)x_n \to (\lambda_0 - A)^{-1} C(\lambda_0 x - y)$. On the other hand, it follows from $A \subset C^{-1}AC$ that $(\lambda_0 - A)^{-1} C(\lambda_0 - A)x_n = Cx_n \to Cx$, and so $(\lambda_0 - A)^{-1} C(\lambda_0 x - y) = Cx$. Therefore $y = C^{-1}ACx = Ax$. ∎

Now combining Theorem 2.1, Lemma 2.2, and Theorem 3.1 below we get

Corollary 2.3. *Let $\overline{R(C)} = X$. Then A generates an almost periodic C-regularized group if and only if the following conditions are satisfied*:

(a) $\overline{D(A)} = X$, $A = C^{-1}AC$, *and* $\mathbf{R} \setminus \{0\} \subset \rho_C(A)$.

(b) $R(C) \subset R((\lambda - A)^n)$, $\forall 0 \neq \lambda \in \mathbf{R}, n \in \mathbf{N}$, *and* $\sup\{\|\lambda^n (\lambda - A)^{-n} C\|; 0 \neq \lambda \in \mathbf{R}, n \in \mathbf{N}\} < \infty$.

(c) $X_a = X$(*equivalently*, $\overline{span}\{x \in D(A);\ Ax = \lambda x$ *for some* $\lambda \in P_\sigma(A)\} = X$).

PROPOSITION 2.4. *Let* $X \not\supset c_0$, $\overline{D(A)} = X, T(\cdot)y \in AP(\mathbf{R}, X)$, $\forall y \in R(A)$, *and* $T(t)$ *be uniformly bounded. Then* $T(t)$ *is almost periodic.*

Proof. By assumption, $M \equiv \sup_{t \in \mathbf{R}} \|T(t)\| < \infty$. Then from (1.3) we have

$$\|\int_0^t T(s)Axds\| = \|T(t)x - Cx\| \le 2M\|x\|, \qquad \forall t \in \mathbf{R}, \qquad x \in D(A).$$

But $T(\cdot)Ax \in AP(\mathbf{R}, X)$; thus $T(\cdot)x \in AP(\mathbf{R}, X)$, $\forall x \in D(A)$ follows from Lemma 1.6(f). Now, for any $x \in X$, by $\overline{D(A)} = X$, there exists $x_n \in D(A)$ such that $\|T(t)x - T(t)x_n\| \le M\|x - x_n\| \to 0$. Hence from $T(\cdot)x_n \in AP(\mathbf{R}, X)$ and Lemma 1.6(g) we deduce $T(\cdot)x \in AP(\mathbf{R}, X)$, $\forall x \in X$. ∎

The following gives the condition that the weak almost periodicity (resp. almost periodicity) of $T(t)$ implies its almost periodicity (resp. uniform almost periodicity).

THEOREM 2.5.

(a) *Suppose that* X *is weakly sequentially complete* (*i.e., every weak Cauchy sequence converges weakly*), *and* $\overline{R(C)} = X$. *Then the weak almost periodicity of* $T(t)$ *implies its almost periodicity.*

(b) *Suppose* $\{e^{\lambda t};\ \lambda \in P_\sigma(A)\}$ *is uniformly almost periodic*; *then the almost periodicity of* $T(t)$ *implies its uniform almost periodicity.*

Proof.

(a) First, it follows from the weak almost periodicity, Lemma 1.6(a) and the uniform boundedness theorem that $T(t)$ is uniformly bounded. Then, by our assumption and Lemma 1.6(c), there exists $\tilde{P}_r x \in X$, such that

$$\langle x^*, \tilde{P}_r x\rangle = \lim_{t\to\infty} \langle x^*, \frac{1}{t}\int_0^t e^{-irs}T(s)xds\rangle, \qquad \forall x \in X, x^* \in X^*, r \in \mathbf{R}. \tag{2.3}$$

It now follows easily from (2.3) that (cf. (2.2))

$$\langle x^*, T(t)\tilde{P}_r x\rangle = \langle x^*, e^{irt}C\tilde{P}_r x\rangle, \qquad \forall t \ge 0, x^* \in X^*.$$

And so $T(t)\tilde{P}_r x = e^{irt}C\tilde{P}_r x$. Thus $\tilde{P}_r x \in \ker(ir - A$, $\forall x \in X, r \in \mathbf{R}$. As seen in the proof of Theorem 2.1 (with $\tilde{P}_r x$ replacing $P_r x$), we deduce $X_a = X$. Therefore, by Theorem 2.1, $T(t)$ is almost periodic.

(b) The uniform boundedness of $T(t)$ follows from the almost periodicity of $T(t)$, which implies $\{T(t)x;\ \|x\| \le 1\}$ is also uniformly bounded. From the proof of the necessity of Theorem 2.1 we see that $\{ir;\ P_r x \neq 0\} \subset P_\alpha(A)$, $\forall x \in X$. Thus the claim now follows from our assumption and [3, Theorem 13]. ▮

3. ALMOST PERIODIC SEMIGROUPS

We first give a basic relationship between regularized semigroups and regularized groups.

THEOREM 3.1. *The following statements are equivalent.*

(a) *A generates a C-regularized group $T(t)$ $(t \in \mathbf{R})$.*

(b) *A and $-A$ generate C-regularized semigroups $T_+(t)$ and $T_-(t)$ $(t \ge 0)$, respectively.*

(c) *A generates a C-regularized semigroup $T_+(t)$, $0 \in \rho_{C^2}(T_+(t))$, $\forall t > 0$, and $T_+(\cdot)^{-1}C^2 x \in C(\mathbf{R}_+, X)$, $\forall x \in X$.*

Furthermore, $T(t) = T_+(t)$ $(t \ge 0)$ and $T(-t) = T_-(t) = T_+(t)^{-1}C^2$ $(t \ge 0)$.

Proof. (a) ⇒ (b). Set $T \pm (t) = T(\pm t)$ $(t \ge 0)$; then both $T_+(t)$ and $T_-(t)$ are C-regularized semigroups. Suppose their generators are A_+ and A_- respectively. By the definition of generators, $\pm A \subset A_\pm$. Conversely, if $x \in D(A_+)$ then $\forall t \in \mathbf{R}$,

$$\lim_{h \downarrow 0} \frac{T(t+h)Cx - T(t)Cx}{h} = T(t) \lim_{h \downarrow 0} \frac{T(h)x - Cx}{h}$$
$$= T(t)CA_+ x \in C(\mathbf{R}, X).$$

By a well-known result (see, e.g., Corollary 2.1.2 in [24], $d/dt\, T(t)Cx = T(t)CA_+x$. Integrating this and operating with C^{-1} on both sides yields $T(t)x = Cx + \int_0^t T(s)A_+x\,ds$ $(t \in \mathbf{R})$. Consequently, $d/dt\, T(t)x\,|_{t=0} = CA_+x \in \mathbf{R}(C)$. In particular, $x \in D(A)$, and so $A_+ = A$. Similarly, $A_- = -A$.

(b) ⇒ (c). For every $x \in X, t \ge 0$, it follows from the assumption and Lemma 1.2(b) and (c) that

$$\frac{d}{dt} T_+(t) \int_0^t T_-(s)x\,ds = T_+(t)A \int_0^t T_-(s)x\,ds + T_+(t)T_-(t)x$$
$$= T_+(t)Cx.$$

Integrating this we get $T_+(t)\int_0^t T_-(s)xds = \int_0^t T_+(s)Cxds$. Now, operating with A on both sides and using (1.2) yields $T_+(t)T_-(t)x = C^2x$. Similarly we can show $T_-(t)T_+(t)x = C^2x$. Thus the claim follows easily from the two formulae.

(c) ⇒ (a). Set $T(t) = T_+(t)$ $(t \geq 0)$ and $T(t) = T_+(-t)^{-1}C^2$ $(t < 0)$. Since by our assumptions $T_+(-t)^{-1}C^2$ is closed, $T_+(-t)^{-1}C^2 \in B(X)$, $\forall t > 0$. Now, it is easy to verify $T(t)$ $(t \in \mathbf{R})$ is a C-regularized group by definition. Suppose its generator is A_0. Then by the proof of (a) ⇒ (b), $A_0 = A$. ∎

If a C-regularized semigroup $T(t)$ satisfies Theorem 3.1(c), then we say the corresponding C-regularized group $T(t)$ exists, where $\hat{T}(t) = T(t)$ $(t \geq 0)$ and $= T(-t)^{-1}C^2$ $(t < 0)$. We now turn to the almost periodicity of $\hat{T}(t)$ and start with the following lemma (see [3]).

LEMMA 3.2.

(a) *Let $S(t)$ be the translation C_0-semigroup on $AP(\mathbf{R}_+, X)$ given by $(S(t)f)(s) = f(t+s)$. Then the corresponding C_0-group $\hat{S}(t)$ exists.*

(b) *Let $E : AP(\mathbf{R}_+, X) \to AP(\mathbf{R}, X)$ be defined by $(Ef)(t) = (\hat{S}(t)f(0), t \in \mathbf{R}$. Then E is a linear surjective isometry and Ef is the unique continuous almost periodic extension of f on $\mathbf{R}$.*

(c) *If $B \in B(X)$ and $f \in AP(\mathbf{J}, X)$, then $Bf \in AP(\mathbf{J}, X)$ and $E(Bg) = BEg$, $\forall g \in AP(\mathbf{R}_+, X)$.*

THEOREM 3.3. *If the C-regularized semigroup $T(t)$ is almost periodic, then the corresponding C-regularized group $\hat{T}(t)$ exists and is also periodic. If, in addition, $T(t)$ is uniformly almost periodic, then so is $\hat{T}(t)$.*

Proof. We first show $T(t)$ is injective, $\forall t > 0$. If $T(t)x = 0$, then for any $\varepsilon > 0$, since $T(\cdot)x \in AP(\mathbf{R}_+, X)$, there exists $\tau > t$, such that $\|T(s+\tau)x - T(s)x\| < \varepsilon$, $\forall s \geq 0$. But $CT(\tau)x = T(\tau - t)T(t)x = 0$ and so $T(\tau)x = 0$. Thus $\|Cx\| = \|T(\tau)x - T(0)x\| < \varepsilon$, which implies $Cx = 0$, i.e., $x = 0$.

Next, for every $x \in X, t > 0$, it follows from Lemma 3.2(c) and the definition of E that

$$\begin{aligned} T(t)\left[(ET_x)(-t)\right] &= \left[E(T(t)T_x)\right](-t) = \left[E(CS(t)T_x)\right](-t) \\ &= C\left[\hat{S}(-t)\hat{S}(t)T_x\right](0) = CT_x(0) = C^2x, \end{aligned}$$

where $T_x = T(\cdot)x$. From this we see $0 \in \rho_{C^2}(T(t))$ and $T(\cdot)^{-1}C^2x = (ET_x)(-\cdot) \in C(\mathbf{R}_+, X)$. Now it follows from Theorem 3.1 and Lemma 3.2(b) that $\hat{T}(t)$ exists and $\hat{T}(t)x = (ET_x)(t)$ $(t \in \mathbf{R})$. From this, it is not difficult to show the claim. ∎

The subsequent theorem contains a characterization and some properties of almost periodic regularized semigroups.

THEOREM 3.4. *Let A generate a C-regularized semigroup $T(t)$ and $\overline{\mathbf{R}(C)} = X$. Then $T(t)$ is almost periodic if and only if $T(t)$ is uniformly bounded and $X_a = X$. In this case, the following are valid.*

(a) *$\sigma_C(A) \subset i\mathbf{R}$. If in addition $P_\sigma(A)$ is bounded, then $A \in B([\mathbf{R}(C)], X)$ where $[\mathbf{R}(C)] = (\mathbf{R}(C), \|C^{-1} \cdot \|)$.*

(b) *If ir is a pole of $R_C(\lambda, A)$, then ir is its simple pole (in particular, $ir \in P_\sigma(A)$), and its residue is P_r defined by (2.1).*

Proof. The proof of sufficiency is the same as that of Theorem 2.1. If $T(t)$ is almost periodic, then by Theorem 3.3 its corresponding C-regularized group $\hat{T}(t)$ is almost periodic, and by Theorem 3.1 its generator is still A. Hence the proof of necessity follows from Theorem 2.1.

Now we show (a). Since A generates the almost periodic C-regularized group $\hat{T}(t)$, $\sigma_C(A) \subset i\mathbf{R}$ follows from Theorem 2.1 and Lemma 1.2(d). To prove the remaining statement in (a) we choose $x \in D(A)$ and $x^* \in X^*$ arbitrarily. By the proof of Theorem 2.1, $P_r x \in \ker(ir - A)$, $\forall r \in \mathbf{R}$. Thus $\{r \in \mathbf{R}; \langle x^*, P_r x\rangle \neq 0\} \subset (1/i) P_\sigma(A)$, but $\langle x^*, P_r x\rangle = a_r(\langle x^*, T(\cdot)x\rangle\}$; hence by our assumption and [11, Theorem 4.8] there exists a constant K (independent of x, x^*) such that

$$|\langle x^*, AT(t)x\rangle| = |\frac{d}{dt}\langle x^*, T(t)x\rangle| \le K \sup_{t\in\mathbf{R}} |\langle x^*, T(t)x\rangle|$$
$$\le KM\|x^*\| \cdot \|x\|,$$

where $M = \sup_{t\in\mathbf{R}}\|T(t)\|$. Set $t = 0$; it then follows that $\|ACx\| \le KM\|x\|$, i.e., $\|Ax\| \le KM\|C^{-1}x\|$, $\forall x \in D(A)$. Also by $\overline{R(C)} = \mathrm{X}$ and Lemma 2.1(a), it is known that $CD(A)$ is dense in $[R(C)]$. Hence $A \in B([R(C)], X)$.

Finally, (b) follows from Lemma 1.6(c), (1.4), and the Abelian theorem of vector-valued functions (see [14, Theorem 18.2.1]). ∎

By Theorem 3.4 and Lemma 2.2, we can characterize the generator of almost periodic regularized semigroups immediately. Moreover, we can replace $\overline{R(C)} = X$ by the weaker condition $\overline{D(A)} = X$ in the second conclusion of Theorem 3.4(a).

4. ALMOST PERIODIC REGULARIZED COSINE FUNCTIONS

In this section we consider the almost periodicity of regularized cosine functions and the corresponding sine functions. Since the methods are essentially the same as in Section 2, we only indicate the various points of the proofs. In the sequel, suppose that $C(t)$ is a C-regularized cosine function with generator A, and $S(t)$ is its corresponding C-regularized sine function.

THEOREM 4.1. *Let* $\overline{\mathbf{R}(C)} = X$. *Then*

(a) *$C(t)$ is almost periodic if and only if $C(t)$ is uniformly bounded and* $X_b = X$.

(b) *$S(t)$ is almost periodic if and only if $S(t)$ is uniformly bounded,* $0 \notin P_\sigma(A)$, *and* $X_b = X$.

Proof.

(a) This is similar to the proof of Theorem 2.1. We only need to note that $C(t)x = \cos(rt)C_x$ for $x \in \ker(r^2 + A)$ $(r \in \mathbf{R})$ in the proof of sufficiency. To show the necessity, we only need to point out that $P_r x \in \ker(r^2 + A)$, where

$$P_r x = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} C(s) x\, ds, \qquad \forall x \in X, r \in \mathbf{R}. \tag{4.1}$$

In fact, by Lemma 1.6(c), the right side of (4.1) exists, and by the definition of C-regularized cosine functions,

$$\begin{aligned} \frac{2}{h^2}(C(h) - C)P_r x &= \frac{1}{h^2} \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} C(C(s+h) \\ &\qquad + C(s-h))x\, ds - \frac{2}{h^2} C P_r x \\ &= \frac{1}{h^2}(e^{irh} + e^{-irh} - 2) C P_r x \to -r^2 C P_r x (h \to 0). \end{aligned} \tag{4.2}$$

Thus the claim follows.

(b) If $S(t)$ is almost periodic, then by Lemma 1.6(a) and the uniform boundedness theorem $S(t)$ is uniformly bounded. Also, from Lemma 1.6(c) we know that

$$P_r x \equiv \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} S(s) x\, ds, \qquad \forall r \in \mathbf{R}, x \in X \tag{4.3}$$

exists. Thus by Lemma 1.4(c) it is easy to see that (4.2) still holds, in which P_r is defined by (4.3). If $x^* \in X^*$ such that $\langle x^*, x\rangle = 0$, $\forall x \in X_b$, then by the proof of the necessity of Theorem 2.1, we see $\langle x^*, S(t)x\rangle = 0$, $\forall t \in$

$\mathbf{R}$, $x \in X$. Differentiating this with respect to t and then letting $t = 0$ yields $\langle x^*, Cx \rangle = 0$, $x \in X$. By virtue of $\overline{\mathbf{R}(C)} = X$, we have $x^* = 0$. Thus $X_b = X$. To show $0 \notin P_\sigma(A)$ we let $Ax = 0$. Then $S(t)x = tCx$, $\forall t \in \mathbf{R}$. Taking now $r = 0$ in (4.3) we get that $P_0 x = \lim_{t \to \infty} 1/t\, Cx$. Since $P_0 x$ exists, $x = 0$.

The converse is similar to the proof of the sufficiency of Theorem 2.1; here we note that $S(t)x = 1/r \sin(rt)Cx$ for $x \in \ker(r^2 + A)$ $(r \in \mathbf{R} \setminus \{0\})$. ∎

Using the following lemma (see Corollary 3.4 in [17] and Corollary 3.9 in [19]), we can characterize, in terms of the generator, the almost periodicity of regularized cosine functions and the uniform boundedness of the corresponding regularized sine functions.

LEMMA 4.2. *Let $\overline{\mathbf{R}(C)} = X$ and $M > 0$. Then*

(a) *A generates a C-regularized cosine function $C(t)$ with $\|C(t)\| \le M$ $(t \in \mathbf{R})$ if and only if $\overline{D(A)} = X$, $A = C^{-1}AC$, $(0, \infty) \subset \rho_C(A)$, and $\|(\lambda R_C(\lambda^2, A))^{(n)}\| \le Mn!\lambda^{-n-1}$, $\forall \lambda > 0, n \in N_0$.*

(b) *If A generates a C-regularized cosine function $C(t)$ $(t \in \mathbf{R})$, then $\|S(t)\| \le M (t \in \mathbf{R})$ if and only if $(0, \infty) \subset \rho_C(A)$ and $\|R_C(\lambda^2, A)^{(n)}\| \le Mn!\lambda^{-n-1}$, $\forall \lambda > 0, n \in N_0$.*

Now we give the relationship between the almost periodicity of $C(t)$ and $S(t)$, and the characterization for both $C(t)$ and $S(t)$ being almost periodic.

THEOREM 4.3. *Let $X \not\supset c_0$.*

(a) *If $C(t)$ is almost periodic and $S(t)$ is uniformly bounded, then $S(t)$ also is almost periodic.*

(b) *If $\overline{D(A)} = X$, $S(\cdot)y \in AP(\mathbf{R}, X)$, $\forall y \in \mathbf{R}(A)$, and $C(t)$ is uniformly bounded, then $C(t)$ is almost periodic.*

(c) *Assume $\overline{\mathbf{R}(C)} = X$. Then both $C(t)$ and $S(t)$ are almost periodic if and only if both $C(t)$ and $S(t)$ are uniformly bounded and $X_b = X$.*

Proof. (a) follows directly from Lemma 1.6(f). By Lemma 1.4 and the same way as in the proof of Proposition 2.4 we can show (b). Finally, (c) follows from (a) and Theorem 4.1(a). ∎

If $x, y \in CD(A)$, then from [15, Theorem 1.5(i)] (also see [17, Theorem 2.3]) we know that the second-order abstract Cauchy problem

$$x''(t) = Ax(t), \qquad x(0) = x, \qquad x'(0) = y \tag{4.4}$$

has a unique solution $x(t) = C(t)C^{-1}x + S(t)C^{-1}y$ $(t \in \mathbf{R})$. Thus the purpose of characterizing the almost periodicity of both $C(t)$ and $S(t)$ is to get almost periodic solutions of (4.4).

Finally, we consider the weak and uniform almost periodicity of $C(t)$ and $S(t)$. The proof of the following results is similar to that of Theorem 2.3 (note also the method used in the proof of Theorem 4.1) and is omitted.

THEOREM 4.4.

(a) *Suppose X is weakly sequentially complete and $\overline{\mathbf{R}(C)} = X$. Then the weak almost periodicity of $C(t)$ (resp. $S(t)$) implies the almost periodicity of $C(t)$ (resp. $S(t)$).*

(b) *Suppose $\{e^{\lambda t}; \lambda^2 \in P_\sigma(A)\}$ is uniformly almost periodic. Then the almost periodicity of $C(t)$ (resp. $S(t)$) implies the uniform almost periodicity of $C(t)$ (resp. $S(t)$).*

5. PERIODIC REGULARIZED GROUPS AND COSINE FUNCTIONS

In view of the remarks at the end of Section 1 we only consider the periodicity of regularized groups, cosine functions, and the corresponding sine functions. We start with the characterization and the properties of periodic regularized groups.

THEOREM 5.1. *Assume that A generates a C-regularized group $T(t)$ and $\overline{\mathbf{R}(C)} = X$. Then $T(t)$ is a periodic C-regularized group with period p if and only if $\sigma_C(A) \subset (2\pi \mathbf{i}/p)Z$ and $X_a = X$. In this case, the following statements are true.*

(a) *$\sigma_C(A)$ consists of all the simple poles of $R_C(\lambda, A)$. In particular $\sigma_C(A) = P_\sigma(A)$.*

(b) *If $\lambda \in \mathbf{C}(2\pi \mathbf{i}/p)Z$, then*

$$R_C(\lambda, A)x = (1 - e^{-\lambda p})^{-1} \int_0^p e^{-\lambda s} T(s) x ds, \qquad \forall x \in X. \quad (5.1)$$

(c) *Let P_k be the residue of $R_C(\lambda, A)$ at $(2\pi \mathbf{i}/p)k$, then*

$$T(t)x = \sum_{k \in \mathbf{Z}} e^{2\pi \mathbf{i} kt/p} P_k x, \qquad \forall x \in D(A), \qquad t \in \mathbf{R} \quad (5.2)$$

and

$$CAx = \sum_{k \in \mathbf{Z}} \frac{2\pi \mathbf{i} k}{p} P_k x, \qquad \forall x \in D(A^2). \quad (5.3)$$

Proof. Necessity. By (1.1) and $T(p) = C$,

$$(\lambda - A)\int_0^p e^{-\lambda s}T(s)x\,ds = (1 - e^{-\lambda p})Cx, \qquad \forall x \in X, \qquad \lambda \in \mathbf{C}. \tag{5.4}$$

Combining this with $T(s)Ax = AT(s)x$, $\forall x \in D(A)$, $s \geq 0$, one obtains (b) and $\sigma_C(A) \subset (2\pi\mathbf{i}/p)\mathbf{Z}$, while $X_a = X$ follows from Theorem 2.1.

Sufficiency. If $x \in \ker(2\pi\mathbf{i}k/p - A)$ $(k \in \mathbf{Z})$, then $T(t)x = e^{2\pi\mathbf{i}kt/p}Cx$. Consequently $T(t+p)x = T(t)x$, $\forall t \in \mathbf{R}$. But by our assumptions, $X = \overline{\operatorname{span}} \cup_{k \in \mathbf{Z}} \ker(2\pi ik/p - A)$, and so $T(t+p) = T(t)$, $\forall t \in \mathbf{R}$.

It now remains to show (a) and (c). By (5.1),

$$\begin{aligned} P_k x &= \lim_{\lambda \to 2\pi\mathbf{i}k/p} (\lambda - 2\pi\mathbf{i}k/p)R_C(\lambda, A)x \\ &= \frac{1}{p}\int_0^p e^{-2\pi\mathbf{i}ks/p}T(s)x\,ds, \qquad \forall x \in X, \qquad k \in \mathbf{Z}. \end{aligned} \tag{5.5}$$

Thus (a) holds and $P_k x k \in \mathbf{Z}$ are the Fourier coefficients of $T(t)x$. Then since $T(\cdot)x \in C^1(\mathbf{R}, X)$, $\forall x \in D(A)$, we know that (5.2) holds. And from (a), $\sigma_C(A) \subset (2\pi\mathbf{i}/p)\mathbf{Z}$, and the definitions of P_k it easily follows that $\mathbf{R}(P_k) \subset \ker(2\pi\mathbf{i}k/p - A)$, $\forall k \in \mathbf{Z}$. Hence combining this with (5.2) (taking $t = 0$), we obtain

$$CAx = \sum_{k \in \mathbf{Z}} P_k Ax = \sum_{k \in \mathbf{Z}} AP_k x = \sum_{k \in \mathbf{Z}} \frac{2\pi\mathbf{i}k}{p} P_k x, \qquad \forall x \in D(A^2). \quad ∎$$

The result obtained in the proof of necessity (except $X_a = X$) and (5.5) had been given by Da Prato [9]. In Theorem 5.1, $X_a = X$ can be replaced by (a) and

$$Cx = \sum_{k \in \mathbf{Z}} P_k x, \qquad \forall x \in D(A). \tag{5.6}$$

In fact, it follows from (5.6) that $\overline{CD(A)} \subset \overline{\operatorname{span}} \cup_{k \in \mathbf{Z}} \mathbf{R}(P_k)$. But $\overline{\mathbf{R}(C)} = X$ implies $\overline{CD(A)} = X$, and thus $\overline{\operatorname{span}} \cup_{k \in \mathbf{Z}} \mathbf{R}(P_k) = X$. On the other hand, by (a) and $\sigma_C(A) \subset (2\pi\mathbf{i}/p)\mathbf{Z}$, it follows that

$$R(P_k) \subset \ker(2\pi\mathbf{i}k/p - A), \forall k \in \mathbf{Z}.$$

Therefore $X_a = X$. The inverse follows immediately from (5.2) with $t = 0$.

The following is about periodic regularized cosine functions.

THEOREM 5.2. *Suppose A generates a C-regularized cosine function $C(t)$ and $\overline{\mathbf{R}(C)} = X$. Then $C(t)$ is periodic with period p if and only if $\sigma_C(A) \subset$*

$\{-4\pi^2k^2/p^2;\ k \in N_0\}$ *and* $X_b = X$. *In this case, the following statements are true.*

(a) $\sigma_C(A)$ *consists of all the simple poles of* $R_C(\lambda, A)$. *In particular* $\sigma_C(A) = P_\sigma(A)$.

(b) *If* $\lambda \in \mathbf{C}(2\pi\mathbf{i}/p)\mathbf{Z}$, *then*

$$\lambda R_C(\lambda^2, A)x = (1 - e^{-\lambda p})^{-1} \int_0^p e^{-\lambda s} C(s)x ds, \qquad \forall x \in X. \quad (5.7)$$

(c) *Let* P_k $(k \in \mathbf{N}_0)$ *be the residue of* $R_C(\lambda, A)$ *at* $4\pi^2k^2/p^2$, *then*

$$C(t)x = \sum_{k=0}^{\infty} \cos\left(\frac{2\pi kt}{p}\right) P_k x, \qquad \forall x \in D(A), \qquad t \in \mathbf{R} \quad (5.8)$$

and

$$CAx = -\sum_{k=0}^{\infty} \frac{4\pi^2k^2}{p^2} P_k x, \qquad \forall x \in D(A^2), \qquad t \in \mathbf{R}. \quad (5.9)$$

Proof. Necessity. For every $\lambda \in \mathbf{C} \setminus (2\pi\mathbf{i}/p)\mathbf{Z}$, $x \in X$, let $F(\lambda)x$ be the right side of (5.7), then for all $t \in \mathbf{R}$,

$$2C(t)F(\lambda)x = \frac{e^{\lambda t}}{1 - e^{-\lambda p}} \int_t^{p+t} e^{-\lambda s} C(s)Cx\, ds + \frac{e^{-\lambda t}}{1 - e^{-\lambda p}} \int_{-t}^{p-t} e^{-\lambda s} C(s)Cx\, ds.$$

Consequently $C''(t)F(\lambda)x = \lambda^2 C(t)F(\lambda)x - \lambda C(t)Cx$. Letting $t = 0$ we see that $AF(\lambda)x = \lambda^2 F(\lambda)x - \lambda CXx$. Then, by Lemma 1.4(e), $F(\lambda) = \lambda R_C(\lambda^2, A)$. Hence (5.7) holds. The claim now follows from Theorem 4.1(a).

Sufficiency. This easily follows from $C(t)x = \cos(2\pi k/pt)Cx$, $\forall x \in \ker(4\pi^2k^2/p_2 + A)$ $(k \in N_0)$ and the assumptions.

By (5.7) we deduce (a) and

$$P_k x = \frac{\delta_k}{p} \int_0^p e^{-2\pi\mathbf{i}ks/p} C(s)x ds = \frac{\delta_k}{p} \int_o^p \cos\left(\frac{2\pi ks}{p}\right) C(s)x\, ds,$$
$$\forall x \in X, k \in N_0, \quad (5.10)$$

where $\delta_k = 1$ $(k = 0)$ and $\delta_k = 2$ $(k \in \mathbf{N})$. Since $C(\cdot)x \in C^1(\mathbf{R}, X)$, $\forall x \in D(A)$ and $C(t) = C(-t)$, $\forall t \in \mathbf{R}$, (5.8) follows. Also $\mathbf{R}(P_k) \subset \ker(4\pi^2k^2/p^2 + A)$ $(k \in \mathbf{N}_0)$ follows from (a) and $\sigma_C(A) \subset \{-4\pi^2k^2/p^2;\ k \in \mathbf{N}_0\}$. Combining this and (5.8) (with $t = 0$) we have (5.9). ∎

Indeed, from the proof of (5.8) we see that (5.8) is valid for $x \in X_1 = \{x \in X;\ C(\cdot)x \in C^1(\mathbf{R}, X)\}$. Hence (5.9) is also valid for $x \in \{x \in D(A);\ Ax \in X_1\}$. Moreover, in Theorem 5.2, $X_b = X$ can be replaced by (a) and $Cx = \sum_{k=0}^{\infty} P_k x$, $\forall x \in D(A)$.

THEOREM 5.3. *Suppose A generates a C-regularized cosine function $C(t)$ and $\overline{\mathbf{R}(C)} = X$. Then the following statements are equivalent.*

(a) *Both $C(t)$ and $S(t)$ are periodic with period p.*

(b) *$S(t)$ is periodic with period p.*

(c) *$C(t)$ is periodic with period p and $0 \in \rho_C(A)$.*

(d) *$\sigma_C(A) \subset \{-4\pi^2 k^2/p^2;\ k \in \mathbf{N}\}$ and $X_b = X$.*

Proof. By Theorem 5.2, we only need to show the equivalence of (b) and (c). If (b) holds, then differentiating $S(t+p)x = S(t)x$, one obtains $C(t+p) = C(t)$, $\forall t \in \mathbf{R}$. To show the remaining statement we integrate both sides of (1.5) from 0 to p and find

$$A \int_0^p (p-s) S(s) x\, ds = S(p)x - pCx = -pCx, \qquad \forall x \in X.$$

Thus $\mathbf{R}(C) \subset \mathbf{R}(A)$. Also, by Theorem 4.1(b), $0 \notin P_\sigma(A)$ and therefore we have $0 \in \rho_C(A)$. If (c) holds, then (d) is also valid by Theorem 5.2. Hence from this and $S(t)x = (p/2\pi k)\sin(2\pi kt/p)\, Cx$, $\forall x \in \ker(4\pi^2 k^2/p^2 + A)$ $(k \in \mathbf{N})$, we get (b) immediately. ∎

Integrating (5.10) by parts and noting that $S(\cdot)x \in C^1(\mathbf{R}, X)$, $\forall x \in X$, we know

$$\begin{aligned} S(t)x &= \sum_{k=1}^{\infty} \frac{p}{2\pi k} \sin\left(\frac{2\pi kt}{p}\right) P_k x \\ &= \frac{2}{p} \sum_{k=1}^{\infty} \sin\left(\frac{2\pi kt}{p}\right) \int_0^p \sin\left(\frac{2\pi ks}{p}\right) S(s) x\, ds, \end{aligned}$$

$$\forall x \in X, \qquad t \in \mathbf{R}.$$

6. A SPECIAL CASE

In this section, we give more properties for almost periodic C-regularized semigroups and cosine functions with $C = R(\lambda_0, A)^n$ $(\lambda_0 \in \rho(A))$. To this end we need the following.

LEMMA 6.1. *Assume A is an operator on X with $\rho(A) \neq \emptyset$. Let $C = R(\lambda_0, A)^n$, where $\lambda_0 \in \rho(A)$, $n \in N_0$. Then*

(a) $\rho(A) = \rho_C(A)$ *and* $\sigma(A) = \sigma_C(A)$.

(b) *If A is densely defined, then* $\overline{\mathbf{R}(C)} = X$.

(c) $R_C(\lambda, A) = R(\lambda, A)C$ *and*

$$R(\lambda, A) = (\lambda_0 - \lambda)^n R_C(\lambda, A) + \sum_{k=1}^{n} (\lambda_0 + \lambda)^{k-1} R(\lambda_0, A)^k, \qquad \forall\lambda \in \rho(A). \quad (6.1)$$

Proof.

(a) If suffices to show $\rho_C(A) \subset \rho(A)$. If $\lambda \in \rho_C(A)$, then $\forall x \in X$,

$$y \equiv R_C(\lambda, A)x = (\lambda - A)^{-1} R(\lambda_0, A)^n x \in D(A^{n+1})$$

and so $x = (\lambda - A)(\lambda_0 - A)^n y \in \mathbf{R}(\lambda - \mathrm{A})$, i.e., $\mathbf{R}(\lambda - A) = X$. On the other hand, $\lambda - A$ is injective and A is closed. There $\lambda \in \rho(A)$.

(b) We only need to note that $\overline{D(A)} = X$ and $\rho(A) \neq \emptyset$ imply $\overline{D(A^n)} = X$.

(c) $R_C(\lambda, A) = R(\lambda, A)C$ ($\lambda \in \rho(A)$) is obvious, while (6.1) follows from the resolvent identity. ∎

In the sequel, let A be a densely defined operator with $\rho(A) \neq \emptyset$. Set $C = R(\lambda_0, A)^n$ ($\lambda_0 \in \rho(A)$, $n \in \mathbf{N}_0$).

THEOREM 6.2. *Suppose A generates an almost periodic C-regularized semigroup $T(t)$, where $C = R(\lambda_0, A)^n$. Let $M_r = (\lambda_0 - ir)^n P_r$ ($r \in \mathbf{R}$), where P_r is defined by* (2.1). *Then*

(a) $\sigma(A) \subset i\mathbf{R}$, *and*

$$\lim_{\lambda \downarrow ir} (\lambda - ir) R(\lambda, A)x = M_r x, \qquad \forall r \in \mathbf{R}, \qquad x \in X, \qquad (6.2)$$

where $\lambda \downarrow ir$ denotes $\lambda = \tau + ir$ and $\tau \downarrow 0$.

(b) *$\forall r \in \mathbf{R}$, M_r is a bounded projection onto $ker(ir - A)$ along $\overline{R(ir - A)}$ (i.e., $M_r^2 = M_r \in B(X)$, $ker(M_r) = \overline{R(ir - A)}$, and $R(M_r) = ker(ir - A)$) and $X = ker(ir - A) \oplus \overline{R(ir - A)}$.*

(c) *If ir ($r \in \mathbf{R}$) is an isolated point of $\sigma(A)$, then ir is the simple pole of $R(\lambda, A)$ (in particular, $ir \in P_\sigma(A)$) and its corresponding spectral projection is M_r.*

Proof.

(a) Since $T(t)$ is uniformly bounded, $\sigma_C(A) \subset i\mathbf{R}$. Then by Lemma 6.1(a), $\sigma(A) \subset i\mathbf{R}$. Thus by (2.1), (1.4) and the Abelian theorem of vector-valued functions, it follows that

$$\lim_{\lambda \downarrow ir} (\lambda - ir) R_C(\lambda, A) x = P_r x, \qquad \forall x \in X, \qquad r \in \mathbf{R}. \tag{6.3}$$

Combining this, $\sigma(A) \subset i\mathbf{R}$, and (6.1) we easily see that (6.2) holds.

(b) For fixed $r \in \mathbf{R}$, set $A_\lambda = (\lambda - ir)R(\lambda, A)$ and $B_\lambda = R(\lambda, A)$ $(\lambda \in \rho(A))$. By (2.1) we get $\|P_r\| \leq M$, where $M = \sup_{t \geq 0}\|T(t)\|$. Thus, by (6.2), $\|A_\lambda\| = O(1)$ $(\lambda \downarrow ir)$, which implies

$$\|(A - ir)A_\lambda\| \leq \|\lambda - ir|(1 + \|A_\lambda\|) \to 0(\lambda \downarrow ir).$$

Also

$$\mathbf{R}(A_\lambda) = \mathbf{R}(B_\lambda) \subset D(A) = D(A - ir), \qquad \forall \lambda \in \rho(A)$$

and

$$B_\lambda(A - ir) \subset (A - ir)B_\lambda = A_\lambda - I, \qquad \forall \lambda \in \rho(A).$$

Hence (b) follows now from (6.2), Theorem 1.1, and its remark in [27].

(c) Let P be the spectral projection associated with A and ir. Then $X = \ker(P) \oplus R(P)$. With respect to this decomposition, we write $A = A_0 \oplus A_1$ and $T(t) = T_0(t) \oplus T_1(t)$, where we note that $T(t)$ and P commute. Then $T_1(t)$ is an almost periodic C_1-regularized semigroup on $\mathbf{R}(P)$ with generator A_1, where $C_1 = R(\lambda_0, A_1)^n$. Hence by Theorem 3.4, the set of eigenvectors of A_1 spans a dense subspace of $\mathbf{R}(P)$. Since $\sigma(A_1) = \{ir\}$, it follows that $\ker(ir - A_1) = \mathbf{R}(P)$, i.e., $A_1 = irI|_{\mathbf{R}(P)}$. Therefore, since $R(\lambda, A) = R(\lambda, A_0) \oplus R(\lambda, A_1)$ and $ir \in \rho(A_0)$, ir is a simple pole of $R(\lambda, A)$. Finally, since

$$\ker(ir - A) = \ker(ir - A_0) \oplus \ker(ir + A_1) = \{0\} \oplus \mathbf{R}(P) = \mathbf{R}(P)$$

and since

$$\mathbf{R}(ir - A) = \mathbf{R}(ir - A_0) \oplus \mathbf{R}(ir - A_1) = \ker(P) \oplus \{0\} = \ker(P)$$

it follows from (b) that $P = M_r$. ∎

Similarly, we can give the following.

THEOREM 6.3. *Let A generate a C-regularized cosine function $C(t)$, where $C = R(\lambda_0, A)^n$, and $C(t)$ (resp. $S(t)$) is almost periodic. Let* $M_r = 2(\lambda_0 +$

$r^2)^n P_r$ (*resp.* $= 2ir(\lambda_0 + r^2)^n P_r$) *for* $r > 0$ *and* $M_0 = \lambda_0 P_0$ (*resp.* $= 0$), *where* P_r *is defined by* (4.1) (*resp*. (4.3)). *Then*

(a) $\sigma(A) \subset (-\infty, 0]$ (*resp.* $\subset (-\infty, 0)$), *and*

$$\lim_{\lambda \downarrow r^2} (\lambda + r^2) R(\lambda, A)x = M_r x, \qquad \forall r \geq 0, \qquad x \in X, \tag{6.4}$$

where $\lambda \downarrow -r^2$ *means* $\lambda = (\tau + ir)^2$ *and* $\tau \downarrow 0$.

(b) *For every* $r \geq 0$, M_r *is a bounded projection onto* $\ker(r^2 + A)$ *along* $\overline{\mathbf{R}(r^2 + A)}$ *and* $X = \ker(r^2 + A) \oplus \overline{\mathbf{R}(r^2 + A)}$.

(c) *If* $-r^2$ *is an isolated point of* $\sigma(A)$, *then* $-r^2$ *is the simple pole of* $R(\lambda, A)$ (*in particular*, $ir \in P_\sigma(A)$) *and its corresponding spectral projection is* M_r.

Proof. (a) By Theorem 4.1(a) (resp. Theorem 4.1(b)) and Lemma 6.1, $\sigma(\mathrm{A}) \subset (-\infty, 0]$. Also by (4.1) (resp. (4.3)), (1.7), and the Abelian theorem of vector-valued functions,

$$\lim_{\lambda \downarrow ir} (\lambda - ir)\lambda R_C(\lambda^2, A)x \left(resp.\ \lim_{\lambda \downarrow ir} (\lambda - ir) R_C(\lambda^2, A)x \right), \qquad \forall x \in X, r \geq 0. \tag{6.5}$$

This and (6.1) (replacing λ by λ^2) easily leads to (6.4).

The proof of (b) and (c) are the same as that of Theorem 6.2(b) and (c), respectively. Finally, we show $0 \in \rho(A)$ when $S(t)$ is almost periodic. Set $A_\lambda = \lambda R(\lambda, A)$ and $B_\lambda = R(\lambda, A)$ ($\lambda > 0$). Then as shown in the proof of Theorem 6.2(b), A_λ and B_λ satisfy (C1)–(C3) in [28]. Also, by (6.1) and (6.5), $\lim_{\lambda \downarrow 0} \lambda R(\lambda^2, A)x = \lambda_0^n P_0 x$ and so $\|\lambda R(\lambda^2, A)\| = O(1)$ ($\lambda \downarrow 0$). Thus $\|\lambda R(\lambda, A)\| \to 0$ ($\lambda \downarrow 0$). Now $X = \ker(A) \oplus \mathbf{R}(A)$ follows from [28 Theorem 1]. On the other hand, by Theorem 4.1(b), $\ker(A) = \{0\}$ and so $X = \mathbf{R}(A)$, i.e., $0 \in \rho(A)$. ∎

From Theorem 4.1, 6.3(a) and [1, p.56] (cf. the proof of [25, Theorem 2]). we know the following results hold.

THEOREM 6.4. *Let* A *generate a* C*-regularized cosine function* $C(t)$, *where* $C = R(\lambda_0, A)^n$. *Then*

(a) $S(t)$ *is almost periodic if and only if* $S(t)$ *is uniformly bounded*, $0 \in \rho(A)$, *and* $X_b = X$.

(b) *Both* $C(t)$ *and* $S(t)$ *are almost periodic if and only if* $C(t)$ *is uniformly bounded*, $0 \in \rho(A)$, *and* $X_b = X$.

For periodic C*-regularized groups and cosine functions, where* $C = R(\lambda_0, A)^n$, $\sigma_C(A)$ *and* $\rho_C(A)$ *in Theorem* 5.1–5.3 *can be replaced by* $\sigma(A)$

and $\rho(A)$, *respectively, and the other conditions and results also can be modified correspondingly. For example, using Theorems* 5.1 *and* 6.2 *and Lemma* 6.1 *and noting* $\tilde{P_k} = (\lambda_0 - ir)^{-n} P_k = R(\lambda_0, A)^n P_k$ *where* $\tilde{P_k}$ *and* P_k *are respectively the residue of* $R_C(\lambda, A)$ *and* $R(\lambda, A)$ *at* $2\pi \mathbf{i} k/p$, *we have*

THEOREM 6.5. *Let* A *generate a* C*-regularized group* $T(t)$, *where* $C = R(\lambda_0, A)^n$. *Then* $T(t)$ *is a periodic* C*-regularized group with period* p *if and only if* $\sigma(A) \subset (2\pi \mathbf{i}/p)\mathbf{Z}$ *and* $X_a = X$. *In this case, the following statements hold.*

(a) $\sigma(A)$ *consists of all the simple poles of* $R(\lambda, A)$. *In particular* $\sigma(A) = P_\sigma(A)$.

(b) *If* $\lambda \in \mathbf{C} \setminus (2\pi \mathbf{i}/p)\mathbf{Z}$, *then*

$$R(\lambda, A)x = (\lambda_0 - \lambda)^n (1 - e^{-\lambda p})^{-1} \int_0^p e^{-\lambda s} T(s) x ds + \sum_{k=1}^{n} (\lambda_0 - \lambda)^{k-1} R(\lambda_0, A)^k, \qquad \forall x \in X.$$

(c) *Let* P_k *be the spectral projection of* $R(\lambda, A)$ *at* $2\pi \mathbf{i} k/p$. *Then*

$$T(t)x = (\lambda_0 - ir)^{-n} \sum_{k \in \mathbf{Z}} e^{2\pi \mathbf{i} kt/p} P_k x, \qquad \forall x \in D(A), \qquad t \in \mathbf{R}$$

and

$$Ax = \sum_{k \in \mathbf{Z}} \frac{2\pi \mathbf{i} k}{p} P_k x, \qquad \forall x \in D(A^2).$$

We conclude this section with an application of Theorem 3.4 to the tempered distribution semigroup (TDSG). A TDSG is a regular distribution semigroup $\mathcal{T}$ (see [20]) with $\mathcal{T} \in B(\mathcal{S}, B(X))$, where $\mathcal{S}$ is the space of rapidly decreasing test functions. A TDSG $\mathcal{T}$ is said to be almost periodic, if $\mathcal{T}(\delta_t * \phi)x \in AP(\mathbf{R}_+, X)$, $\forall x \in X$, $\phi \in \mathcal{S}$, where δ_t denotes the Dirac measure centered at t.

THEOREM 6.6. *Let* A *be a densely defined operator on* X. *Then* A *generates on almost periodic TDSG* $\mathcal{T}$ *if and only if* $(0, \infty) \subset \rho(A)$, *there exists* $k \in N$, $\lambda_0 \in \rho(A)$, *and* $M > 0$ *such that*

$$\| \lambda^n R(\lambda, A)^n R(\lambda_0, A)^k \| \le M, \qquad \forall \lambda > 0, \qquad n \in N, \tag{6.6}$$

and $X_a = X$.

Proof. Necessity. It follows from [8] that A also generates a C-regularized semigroup $T(t) \equiv \mathscr{T}(\delta_t)C$, where $C = R(\lambda_0, A)^k$ for some $k \in N$ and $\lambda_0 > 0$. Let $\phi_0(t) = (t^{k-1}/(k-1)!)e^{-\lambda_0 s}$ and $Y(t)$ be the Heaviside function. Then $T(t)\mathscr{T}(\delta_t)\mathscr{T}(Y\phi_0) = \mathscr{T}(\delta_t * (Y\phi_0))$. Thus, by our definitions, $T(t)$ is an almost periodic C-regularized semigroup with the generator A. The claim follows now from Theorem 3.4 and Lemmas 2.2 and 6.1.

Sufficiency. By (6.6) and Lemma 2.2 we know that A generates a uniformly bounded C-regularized semigroup $T(t)$, where $C = R(\lambda_0, A)^k$. It thus follows from Lemma 1.2(d) and 6.1(a) that $\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > 0\} \subset \rho(A)$ and $\|R(\lambda, A)C\| \leq M/\operatorname{Re}\lambda$ for $\operatorname{Re}\lambda > 0$. Using Lemma 6.1(c) again, we get that $\|R(\lambda, A)\| \leq M(1 + |\lambda|)^k/\operatorname{Re}\lambda$ for $\operatorname{Re}\lambda > 0$, which implies that A generates a TDSG $\mathscr{T}$ (cf. [5]), and so, a C_1-regularized semigroup $T_1(t) = \mathscr{T}(\delta_t)C_1$, where $C_1 = R(\lambda_1, A)^{k+2}(\lambda_1 > 0)$. Since $T(t)C^{-1}C_1$ is also a C_1-regularized semigroup with the generator $A, T_1(t) = T(t)C^{-1}C_1$. In particular, $T_1(t)$ is uniformly bounded, and thus, by $X_a = X$, almost periodic. Also, $\forall x \in X$, $\phi \in \mathscr{S}$, by $A = \overline{\mathscr{T}(-\delta_0')}$ we see $\mathscr{T}(\phi)x \in D(A^{k+2} = \mathbf{R}(C_1)$. Therefore $\mathscr{T}(\delta_t * \phi)x = T_1(t)(C_1^{-1}\mathscr{T}(\phi)x) \in AP(\mathbf{R}_+, X)$. ∎

REFERENCES

1. M. Amerio and G. Prouse, "Almost-Periodic Functions and Functional Equations," Van Nostrand–Reinhold, New York, 1971.
2. H. Bart, Periodic strongly continuous semigroups, *Ann. Mat. Pura Appl.* **115** (1977), 311–318.
3. H. Bart and S. Goldberg, Characterizations of almost periodic strongly continuous groups and semigroups, *Math. Ann.* **236** (1978), 105–116.
4. A. G. Baskakov, Harmonic analysis of cosine and exponential operator functions, *Math. USSR-Sb.* **52** (1985), 63–90.
5. I. Cioranescu, Analytic generator and spectral subspaces for tempered distribution groups, *An. Univ. Craiova* **5** (1977), 11–26.
6. I. Cioranescu, On periodic distribution groups, *in* "Topics in Modern Operator Theory" (C. Apostol and R. G. Douglas, Eds.), Birkhäuser, Basel, 1981.
7. I. Cioranescu, Characterizations of almost periodic strongly continuous cosine operator functions, *J. Math. Anal. Appl.* **57** (1986), 222–229.
8. G. Da Prato, Semigruppi regolarizzabili, *Ricerche Mat.* **15** (1966), 223–248.
9. G. Da Prato, Semigruppi periodici, *Ann. Mat. Pura. Appl.* **78** (1968), 55–67.
10. R. deLaubenfels, "Existence Families, Functional Calculi and Evolution Equations," Lecture Notes in Math., Vol. 1570, Springer-Verlag, Berlin, 1994.
11. A. M. Fink, "Almost Periodic Differential Equations," Lecture Notes in Math., Vol. 377, Springer-Verlag, Berlin, 1974.
12. E. Giusti, Funzioni coneno periodische, *Boll. Un. Mat. Ital.* **22** (1967), 478–485.
13. M. Hieber, A. Holderrieth, and F. Neubrander, Regularized semigroups and systems of linear partial differential equations, *Ann. Scuola Norm. di Pisa* **19** (1992), 263–379.

14. E. Hille and R. S. Phillips, "Functional Analysis and Semigroups," Amer. Math. Soc. Colloq. Publ., Vol. 31, Providence, RI, 1957.
15. C.-C. Kuo and S.-Y. Shaw, C-cosine functions and the abstract Cauchy problem I, Preprint, 1993.
16. Y. Lei and Q. Zheng, The application of C-semigroups to differential operators in $L^p(R^n)$, *J. Math. Anal. Appl.* **188**, (1994), 809–818.
17. Y. Lei and Q. Zheng, Exponentially bounded C-cosine functions of operators, *Sys. Sci. Math. Scis.* (to appear). [in Chinese]
18. M. Levitan and V. V. Zhikov, "Almost Periodic Functions and Differential Equations," Cambridge Univ. Press, London, 1982.
19. Y.-C. Li and S.-Y. Shaw, Integrated C-cosine functions and the abstract Cauchy problem, Preprint, 1991.
20. J.-L. Lions, Les semigroupes distributions, *Portugal. Math.* **19** (1960), 141–164.
21. D. Lutz, Periodische operatorwertige cosinusfunktionen, *Resultata Math.* **4** (1981), 75–83.
22. I. Miyadera, A generalization of the Hille-Yosida theorem, *Proc. Japan Acad.* **64** (1988), 223–226.
23. R. Nagel (Ed.), "One-Parameter Semigroups of Positive Operators," Lecture Notes in Math., Vol. 1184, Springer-Verlag, Berlin, 1986.
24. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
25. S. I. Piskarev, Periodic and almost periodic cosine operator functions, *Math. USSR-Sb.* **46** (1982), 391–402.
26. S. I. Piskarev, Almost periodic solutions of second-order differential equations, *Siberian Math. J.* **25** (1984), 451–460.
27. S.-Y. Shaw, Mean ergodic theorems and linear functional equations, *J. Funct. Anal.* **87** (1989), 428–441.
28. S.-Y. Shaw, Uniform convergence of ergodic limits and approximate solutions, *Proc. Amer. Math. Soc.* **114** (1992) 405–411.
29. V. V. Vasilev, S. G. Krein, and S. I. Piskarev, Semigroups of operators, cosine operator functions, and linear differential equations, *J. Soviet Math.* **54** (1991), 1024–1129.
30. S. Zaidman, "Almost-Periodic Functions in Abstract Spaces," Pitman Research Notes in Math., Vol. 126, Pitman, Boston, 1985.